

# Scatter and regularity imply Benford's Law... and more

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## Abstract

A random variable (r.v.)  $X$  is said to follow Benford's law if  $\log(X)$  is uniform mod 1. Many experimental data sets prove to follow an approximate version of it, and so do many mathematical series and continuous random variables. This phenomenon received some interest, and several explanations have been put forward. Most of them focus on specific data, depending on strong assumptions, often linked with the log function.

Some authors hinted - implicitly - that the two most important characteristics of a random variable when it comes to Benford are regularity and scatter.

In a first part, we prove two theorems, making up a formal version of this intuition: scattered and regular r.v.'s do approximately follow Benford's law. The proofs only need simple mathematical tools, making the analysis easy. Previous explanations thus become corollaries of a more general and simpler one.

These results suggest that Benford's law does not depend on properties linked with the log function. We thus propose and test a general version of the Benford's law. The success of these tests may be viewed as an *a posteriori* validation of the analysis formulated in the first part.

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## 1. Introduction

First noticed by Newcomb (1881), and again later by Benford (1938), the so-called Benford's law states that a sequence of "random" numbers should be such that their logarithms are uniform mod 1. As a consequence, the first non-zero digit of a sequence of "random" numbers is  $d$  with probability  $\log(1 + \frac{1}{d})$ , an unexpectedly non-uniform probability law.  $\log$  here stands for the base 10 logarithm, but an easy generalisation follows: a random variable (r.v.) conforms to *base b* Benford's law if its base  $b$  logarithm  $\log_b(X)$  is uniform mod 1. Lolbert (2008) recently proved that no r.v. follows base  $b$  Benford's law for all  $b$ .

Many experimental data roughly conform to Benford's law (most of which no more than roughly). However, the vast majority of real data sets that have been tested do not fit this law at all. For instance, Scott and Fasli (2001) reported that only 12.6% of 230 real data sets passed the test for Benford's law. In his seminal paper, Benford (1938) tested 20 data sets including lakes areas, length of rivers, populations, etc., half of which did not conform to Benford's law.

The same is true of mathematical sequences or continuous r.v.'s. For example, binomial arrays  $\binom{n}{k}$ , with  $n \geq 1$ ,  $k \in \{0, \dots, n\}$ , tend toward Benford's law (Diaconis, 1977), whereas simple sequences such as  $(10^n)_{n \in \mathbb{N}}$  obviously don't.

In spite of all this, Benford's law is actually used in the so-called "digital analysis" to detect anomalies in pricing (Sehity et al., 2005) or frauds, for instance in accounting reports (Drake and Nigrini, 2000) or campaign finance (Cho and Gaines, 2007). Faked data indeed usually depart from Benford's law more than real ones (Hill, 1988). However, Hales et al. (2008) advise caution, arguing that real data do not always fit the law.

Many explanations have been put forward to elucidate the appearance of Benford's law on natural or mathematical data. Some authors focus on particular random variables (Engel and Leuenberger, 2003), sequence (Jolissaint, 2005), real data (Burke and Kincanon, 1991), or orbits of dynamical systems (Berger et al., 2004). As a rule, other explanations assume special properties of the data. Hill (1995b) or Pinkham (1961) shows that scale invariance implies Benford's law. Base invariance is an other sufficient condition (Hill, 1995a). Mixtures of uniform distributions (Janvresse and Delarue, 2004)

also conform to Benford's law, and so do the limits of some random processes (Shürger, 2008). Multiplicative processes have been mentioned as well (Pietronero et al., 2001). Each of these explanations accounts for some appearances of data fitting Benford's law, but lacks generality.

While looking for a truly general explanation, some authors noticed that data sets are more likely to fit Benford's law if they were scattered enough. More precisely, a sequence should "cover several orders of magnitude", as Raimi (1976) expressed it. Of course, scatter alone is no sufficient condition. The sequence 0.9, 9, 90, 900... indeed covers several orders of magnitude, but is far from conforming to Benford's law. The continuous random variables that are known to fit Benford's law usually present some "regularity": exponential densities, normal densities, or lognormal densities are of this kind. Invariance assumptions (base-invariance or scale-invariance) lead to "regular" densities and so do central limit-like theorem assumptions of mixture.

Some technical explanations may be viewed as a mathematical expression of the idea that a random variable  $X$  is more likely to conform to Benford's law if it is regular and scattered enough. (Mardia and Jupp, 2000, Example 4.1.) linked Benford's law to Poincaré's theorem in circular statistics, and Smith (2007) expressed it in terms of Fourier transforms and signal processing. However, a non expert reader would hardly notice the smooth-and-scattered implications of these developments.

Though scatter has been explicitly mentioned and regularity allusively evoked, the idea that scatter and regularity (in a sense that will be made clear further) may actually be a *sufficient* explanation for Benford's phenomenon related to continuous r.v.'s have never been formalized in a simple way, to our knowledge, except in a recent article by Fewster (2009). In this paper, Fewster hypothesizes that "*any distribution [...] that is reasonably smooth and covers several orders of magnitude is almost guaranteed to obey Benford's law.*" He then defines a smoothing procedure for a r.v.  $X$  based on  $[\pi^2(x)]''$ ,  $\pi$  being the probability density function (henceforth *p.d.f.*) of  $\log(X)$ , and illustrates with a few eloquent examples that under smoothness and scatter constraints, a r.v. cannot depart much from Benford's law. However, no theorem is given that would formalise this idea.

In the first part of this paper, we prove a theorem from which it follows that scatter and regularity can be modelled in such a way that they, alone, imply *rough* compliance to Benford's law (again: real data usually do not perfectly fit Benford's law, irrespective of the sample size).

It is not surprising that many data sets or random variables samples are

scattered and regular hence our explanation of Benford's phenomena corroborates a widespread intuition. The proof of this theorem is straightforward and requires only basic mathematical tools. Furthermore, as we shall see, several of the existing explanations can be understood as corollaries of ours. Our explanation encompasses more specific ones, and is far simpler to understand and to prove.

Scatter and regularity do not presuppose any log-related properties (such as the property of log-normality, scale-invariance, or multiplicative properties). For this reason, if we are right, Benford's law should also admit other versions. We set that a r.v.  $X$  is  $u$ -Benford for a function  $u$  if  $u(X)$  is uniform mod 1. The classical Benford's law is thus a special case of  $u$ -Benford's law, with  $u = \log$ . We test real data sets and mathematical sequences for " $u$ -Benfordness" with various  $u$ , and test a second theorem echoing the first one. Most data conform to  $u$ -Benford's law for different  $u$ , which is an argument in favour of our explanation.

## 2. Scatter and regularity: a key to Benford

The basic idea at the root of theorem 1 (below) is twofold.

First, we hypothesize that a continuous r.v.  $X$  with density  $f$  is almost uniform mod 1 as soon as it is scattered and regular. More precisely, any  $f$  that is non-decreasing on  $] -\infty, a]$ , and then non-increasing on  $[a, +\infty[$  (for regularity) and such that its maximum  $m = \sup(f)$  is "small" (for scatter) should correspond to a r.v.  $X$  approaching uniformity mod 1. Figure 1 illustrates this idea.

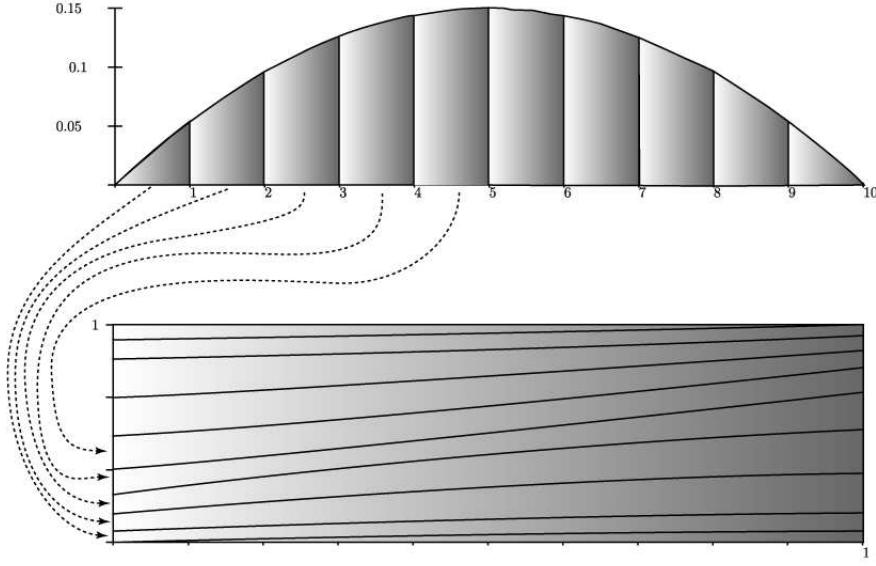


Figure 1 — Illustration of the idea that a regular p.d.f. is almost bound to give rise to uniformity mod 1. The stripes — restrictions of the density on  $[n, n + 1]$  — of the p.d.f. of a r.v.  $X$  are stacked to form the p.d.f. of  $X$  mod 1. The slopes partly compensate, so that the resulting p.d.f. is almost uniform. If the initial p.d.f. is linear on every  $[n, n + 1]$ , the compensation is perfect.

Second, note that if  $X$  is scattered and regular enough, so should be  $\log(X)$ . These two ideas are formalized and proved in theorem 1.

Henceforth, for any real number  $x$ ,  $\lfloor x \rfloor$  will denote the greatest integer not exceeding  $x$ , and  $\{x\} = x - \lfloor x \rfloor$ . Any positive  $x$  can be written as a product  $x = 10^{\lfloor \log x \rfloor} \cdot 10^{\{\log x\}}$ , and the Benford's law may be rephrased as the uniformity of the random variable  $\{\log(X)\}$ .

**Theorem 1.** *Let  $X$  be a continuous positive random variable with p.d.f.  $f$  such that  $\text{Id.f} : x \mapsto xf(x)$  conforms to the two following conditions :  $\exists a > 0$  such that (1)  $\max(\text{Id.f}) = m = a.f(a)$  and (2)  $\text{Id.f}$  is nondecreasing on  $]0, a]$ , and nonincreasing on  $[a, +\infty[$ . Then, for any  $z \in ]0, 1[$ ,*

$$|P(\{\log X\} < z) - z| < 2 \ln(10) m.$$

*In particular,  $(X_n)$  being a sequence of continuous r.v.'s with p.d.f.  $f_n$  satisfying these conditions and such that  $m_n = \max(\text{Id.f}_n) \rightarrow 0$ ,  $\{\log(X_n)\}$  converges toward uniformity on  $[0, 1[$  in law.*

PROOF. We first prove that for any continuous r.v.  $Y$  with density  $g$  such that  $g$  is nondecreasing on  $]-\infty, b]$ , and then nonincreasing on  $[b, +\infty[$ , the following holds:

$$\forall z \in ]0, 1], |P(\{Y\} < z) - z| < 2M$$

where  $M = g(b) = \sup(g)$ .

We may suppose without loss of generality that  $b \in [0, 1[$ . Let  $z \in ]0, 1[$  (the case  $z = 0$  is obvious). Put  $I_{n,z} = [n, n+z]$ . For any integer  $n \leq -1$ ,

$$\frac{1}{z} \int_{I_{n,z}} g(t) dt \leq \int_n^{n+1} g(t) dt.$$

Thus

$$\frac{1}{z} \sum_{n \leq -1} \int_{I_{n,z}} g(t) dt \leq \int_{-\infty}^0 g(t) dt.$$

For any integer  $n \geq 2$ ,

$$\frac{1}{z} \int_{I_{n,z}} g(t) dt \leq \int_{n-1+z}^{n+z} g(t) dt,$$

so

$$\frac{1}{z} \sum_{n \geq 2} \int_{I_{n,z}} g(t) dt \leq \int_{1+z}^{+\infty} g(t) dt.$$

Moreover,  $\int_{I_{0,z}} g \leq zM$  and  $\int_{I_{1,z}} g \leq zM$ . Hence,

$$\frac{1}{z} \sum_{n \in \mathbb{Z}} \int_{I_{n,z}} g \leq \int_{-\infty}^{\infty} g + 2M.$$

We prove in the same fashion that

$$\frac{1}{z} \sum_{n \in \mathbb{Z}} \int_{I_{n,z}} g \geq \int_{-\infty}^{\infty} g - 2M.$$

Since  $\sum_{n \in \mathbb{Z}} \int_{I_{n,z}} g = P(\{Y\} < z)$ ,  $z < 1$  and  $\int_{-\infty}^{\infty} g = 1$ , the result is proved.

Now, applying this to  $Y = \log(X)$  proves theorem 1.

**Remark 1.** The convergence theorem is still valid if we accept  $f$  to have a finite number of monotony changes, provided this number does not exceed a previously fixed  $k$ . The proof is straightforward.

**Remark 2.** The assumptions made on  $Id.f$  may be seen as a measure of scatter and regularity for  $X$ , adjusted for our purpose.

### 3. Examples

#### 3.1. Type I Pareto

A continuous r.v.  $X$  is type I Pareto with parameters  $\alpha$  and  $x_0$  ( $\alpha, x_0 \in \mathbb{R}_+^*$ ) iff it admits a density function

$$f_{x_0, \alpha}(x) = \frac{\alpha x_0^\alpha}{x^{\alpha+1}} \mathbb{I}_{[x_0, +\infty[}$$

Besides its classical use in income and wealth modelling, type I Pareto variables arise in hydrology and astronomy (Paoletta, 2006, page 252).

The function  $Id.f = g : x \mapsto \frac{\alpha x_0^\alpha}{x^\alpha} \mathbb{I}_{[x_0, +\infty[}$  is decreasing. Its maximum is

$$\sup(Id.f) = Id.f(x_0) = \alpha.$$

Therefore,  $X$  is nearly Benford-like, in the extent that

$$|P(\{\log X\} < z) - z| < 2 \ln(10)\alpha.$$

#### 3.2. Type II Pareto

A r.v.  $X$  is type II Pareto with parameter  $b > 0$  iff it admits a density function defined by

$$f_b(x) = \frac{b}{(1+x)^{b+1}} \mathbb{I}_{[0, +\infty[}$$

It arises in a so-called *mixture model*, with mixing components being gamma distributed r.v.'s sequences.

The function  $Id.f_b = g_b : x \mapsto \frac{bx}{(1+x)^{b+1}} \mathbb{I}_{[0, +\infty[}$  is  $C^\infty(\mathbb{R}_+)$ , with derivative

$$g'_b(x) = \frac{b(1-bx)}{(x+1)^{b+2}},$$

which is positive whenever  $x < \frac{1}{b}$ , then negative. From this result we derive

$$\sup g_b = g_b\left(\frac{1}{b}\right) = \frac{1}{\left(1+\frac{1}{b}\right)^{1+b}} = \left(\frac{b}{1+b}\right)^{b+1},$$

since

$$\ln \left[ \left( \frac{b}{1+b} \right)^{b+1} \right] = (b+1) [\ln b - \ln(b+1)],$$

which tends toward  $-\infty$  when  $b$  tends toward 0,

$$\sup g_b \xrightarrow[b \rightarrow 0]{} 0.$$

Theorem 1 applies. It follows that  $X$  conform toward Benford's law when  $b \rightarrow 0$ .

### 3.3. Lognormal distributions

A r.v.  $X$  is lognormal iff  $\log(X) \sim N(\mu, \sigma^2)$ . Lognormal distributions have been related to Benford (?). It is easy to prove that whenever  $\sigma \rightarrow \infty$ ,  $X$  tends toward Benford's law. Although the proof may use different tools, a straightforward way to do it is theorem 1.

One classical explanation of Benford's law is that many data sets are actually built through multiplicative processes (Pietronero et al., 2001). Thus, data may be seen as a product of many small effects. This may be modelled by a r.v.  $X$  that may be written as

$$X = \prod_i Y_i,$$

$Y_i$  being a sequence of random variables. Using the log transformation, this leads to  $\log(X) = \sum \log(Y_i)$ .

The *multiplicative central-limit theorem* therefore proves that, under usual assumptions,  $X$  is bound to be almost lognormal, with  $\log(X) \sim N(\mu, \sigma^2)$ , and  $\sigma \rightarrow \infty$ , thus roughly conforming to Benford, as an application of theorem 1.

## 4. Generalizing Benford

If we are right to think that Benford's law is to be understood as a consequence of mere scatter and regularity, instead of special characteristics linked with multiplicative, scale-invariance, or whatever log-related properties, we should be able to state, prove, and check on real data sets, a generalized version of the Benford's law were some function  $u$  replaces the log.

Indeed, our basic idea is that  $X$  being scattered and regular enough implies  $\log(X)$  to be scattered and regular as well, so that  $\log(X)$  should be almost uniform mod 1. The same should be true of any  $u(X)$ ,  $u$  being a function preserving scatter and regularity. Actually, some  $u$  should even be better shots than log, since log reduces scatter on  $[1, +\infty[$ .

First, let us set out a generalized version of theorem 1, the proof of which is closely similar to that of theorem 1.

**Theorem 2.** *Let  $X$  be a r.v. taking values in a real interval  $I$ , with p.d.f.  $f$ . Let  $u$  be a  $C^1$  increasing function  $I \rightarrow \mathbb{R}$ , such that  $\frac{f}{u'} : x \mapsto \frac{f(x)}{u'(x)}$  conforms to the following:  $\exists a > 0$  such that (1)  $\max\left(\frac{f}{u'}\right) = m = \frac{f}{u'}(a)$  and*

(2)  $\frac{f}{u'}$  is non-decreasing on  $]0, a]$ , and non-increasing on  $[a, +\infty[ \cap I$ . Then, for all  $z \in [0, 1[$ ,

$$|P(\{u(X)\} < z) - z| < 2m.$$

In particular, if  $(X_n)$  is a sequence of such r.v.'s with p.d.f.  $f_n$  and  $\max(f_n/u') = m_n$ , and  $\lim_{n \rightarrow \infty} (m_n) = 0$ , then  $\{u(X_n)\}$  converges in law toward  $U([0, 1[)$  when  $n \rightarrow \infty$ .

A r.v.  $X$  such that  $\{u(X)\} \sim U([0, 1[)$  will be said  $u$ -Benford henceforth.

#### 4.1. Sequence

Although our two theorems only apply to continuous r.v.'s, the underlying intuition that log-Benford's law is only a special case (having, however, a special interest thanks to its implication in terms of leading-digits interpretation) of a more general law does also apply to sequence. In this section, we experimentally test  $u$ -Benfordness for a few sequences  $(v_n)$  and a four functions  $u$ .

We will use six mathematical sequences. Three of them, namely  $(\pi n)_{n \in \mathbb{N}}$ , prime numbers  $(p_n)$ , and  $(\sqrt{n})_{n \in \mathbb{N}}$  are known not to follow Benford. The three others,  $(n^n)_{n \in \mathbb{N}}$ ,  $(n!)_{n \in \mathbb{N}}$  and  $(e^n)_{n \in \mathbb{N}}$  conform to Benford.

As for  $u$ , we will focus on four cases:

$$\begin{aligned} x &\longmapsto \log[\log(x)] \\ x &\longmapsto \log(x) \\ x &\longmapsto \sqrt{x} \\ x &\longmapsto \pi x^2 \end{aligned}$$

The first one increases very slowly, so we may expect that it will not work perfectly. The second leads to the classical Benford's law. The  $\pi$  coefficient of the last  $u$  allows us to use integer numbers, for which  $\{x^2\}$  is nil.

The result of the experiment is given in Table 1.

$v_n$	$\log \circ \log(v_n)$	$\log(v_n)$	$\sqrt{v_n}$	$\pi v_n^2$
$\sqrt{n}$ ( $N = 10\,000$ )	68.90 (.000)	45.90 (.000)	4.94 (.000)	0.02 (.000)
$\pi n$ ( $N = 10\,000$ )	44.08 (.000)	26.05 (.000)	0.19 (1.000)	0.80 (.544)
$p_n$ ( $N = 10\,000$ )	53.92 (.000)	22.01 (0.000)	0.44 (0, 990)	0.69 (.719)
$e^n$ ( $N = 1\,000$ )	6.91 (0.000)	0.76 (1.000)	0.63 (.815)	0.79 (.560)
$n!$ ( $N = 1\,000$ ) <sup>(*)</sup>	7.39 (.000)	0.58 (.887)	0.61 (.844)	0.90 (.387)
$n^n$ ( $N = 1\,000$ ) <sup>(*)</sup>	7.45 (.000)	0.80 (.543)	16.32 (.000)	0.74 (.646)

Table 1 — Results of the Kolmogorov-Smirnov tests applied on  $\{u(v_n)\}$ , with four different functions  $u$  (columns) and six sequences (lines). Each sequence is tested through its first  $N$  terms (from  $n = 1$  to  $n = N$ ), with an exception for  $\log \circ \log(n^n)$  and  $\log \circ \log(n!)$ , for which  $n = 1$  is not considered. Each cell displays the Kolmogorov-Smirnov  $z$  and the corresponding  $p$  value.

The sequences have been arranged according to the speed with which it converges to  $+\infty$  (and so are the functions  $u$ ). None of the six sequences is  $\log \circ \log$ -Benford (but a faster divergent sequence such as  $(10^{e^n})$  would do). Only the last three are  $\log$ -Benford. These are the sequences going to  $\infty$  faster than any polynomial. Only one sequence  $(n^n)$  does not satisfy  $\sqrt{\cdot}$ -Benfordness. However, this can be understood as a pathological case, since  $\sqrt{n^n}$  is integer whenever  $n$  is even, or is a perfect square. Doing the same Kolmogorov-Smirnov test with odd numbers not being perfect squares gives  $z = 0,45$  and  $p = 0,987$ , showing no discrepancy with  $\sqrt{\cdot}$ -Benfordness for  $(n^n)$ . All six sequences are  $\pi^2$ -Benford.

Putting aside the case of  $\sqrt{n^n}$ , what Table 1 reveals is that the convergence speed of  $u(v_n)$  completely determines the  $u$ -Benfordness of  $(v_n)$ . More precisely, it seems that  $(v_n)$  is  $u$ -Benford whenever  $u(v_n)$  increases as fast as  $\sqrt{n}$ , and is not  $u$ -Benford whenever  $u(v_n)$  increase as slowly as  $\ln(n)$ . Of course, this rule-of-thumb is not to be taken as a theorem. Obviously enough, one can actually decide to increase or decrease convergence speed of  $u(v_n)$  without changing  $\{u(v_n)\}$ , adding or subtracting *ad hoc* integer numbers.

Nevertheless, this observation suggests that we give a closer look at sequence  $f(n)$ , where  $f$  is an increasing and concave real function converging toward  $\infty$ , and look for a condition for  $(\{f(n)\})_n$  to converge to uniformity. An intuitive idea is that  $(\{f(n)\})_n$  will depart from uniformity if it does not increase fast enough: we may define brackets of integers — namely  $[f^{-1}(n), f^{-1}(n+1) - 1] \cap \mathbb{N}$ , within which  $\lfloor f(n) \rfloor$  is constant, and of course  $\{f(n)\}$  increasing. If these brackets are "too large", the relative height of the last considered bracket is so important that it overcomes the first terms of the sequence  $f(0), \dots, f(n) \bmod 1$ . In that case, there is no limit to the probability distribution of  $(\{f(n)\})$ . The weight of the brackets should therefore be small relative to  $f^{-1}(n)$ , which may be written as

$$\frac{f^{-1}(n) - f^{-1}(n+1)}{f^{-1}(n)} \xrightarrow{\infty} 0.$$

Provided that  $f$  is regular, this leads to

$$\frac{(f^{-1})'(x)}{f^{-1}(x)} \xrightarrow{\infty} 0,$$

or

$$[\ln(f^{-1}(x))]' \xrightarrow{\infty} 0.$$

Functions  $f : x \mapsto x^\alpha$ ,  $\alpha > 0$  satisfy this condition. Any  $n^\alpha$  should then show a uniform limit probability law, except for pathological cases ( $\alpha \in \mathbb{Q}$ ). Taking  $\alpha = \frac{1}{\pi}$  gives (with  $N = 1000$ ), a Kolmogorov-Smirnov  $z = 1,331$ , and a  $p$ -value 0.058, which means there is no significant discrepancy from uniformity. On the other hand, the log function which does not conform to this condition is such that  $\{\log(n)\}$  is not uniform, confirming once again our rule-of-thumb conjecture.

#### 4.2. Real data

We test three data sets for  $u$ -Benfordness using a Kolmogorov-Smirnov test for uniformity. First data set is the opening value of the Dow Jones, the first day of each month from October 1928 to November 2007. The second and third are country areas expressed in millions of square-km<sup>2</sup> and the populations of the different countries, as estimated in 2008, expressed in millions of inhabitants. The two last sequences are provided by the CIA<sup>1</sup>. Table 2 displays the results.

	$\log \circ \log(v_n)$	$\log(v_n)$	$\sqrt{v_n}$	$\pi v_n^2$
Dow Jones ( $N = 950$ )	5.90 (.000)	5.20 (.000)	0.75 (.635)	0.44 (.992)
Area pays ( $N = 256$ )	1.94 (.001)	0.51 (.959)	0.89 (.404)	1,88 (.002)
Populations ( $N = 242$ )	3.39 (.000)	0.79 (.568)	0.83 (.494)	0.42 (.994)

Table 2 — Results of the Kolmogorov-Smirnov tests applied on  $\{u(v_n)\}$ .

This table confirms our analysis: classical Benfordness is actually less often borne out than  $\sqrt{\cdot}$ -Benfordness on these data. The last column shows that our previous conjectured rule has exceptions: divergence speed is not an absolute criterion by itself. For country areas, the fast growing  $u : x \mapsto \pi x^2$  gives a discrepancy from uniformity, whereas the slow-growing log does not. However, allowing for exceptions, it is still a good rule-of-thumb.

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<sup>1</sup><http://www.cia.gov/library/publications/the-world-factbook/docs/rankorderguide.html>

### 4.3. Continuous r.v.'s

Our theorems apply on continuous r.v.'s. We now focus on three examples of such r.v.'s, with the same  $u$  as above (except for  $\log \circ \log$ , which is not defined everywhere on  $\mathbb{R}_+^*$ ): the uniform density on  $]0, k]$  ( $k > 0$ ), exponential density, and absolute value of a normal distribution.

#### 4.3.1. Uniform r.v.'s

It is a known fact that a uniform distribution  $X_k$  on  $]0, k]$  ( $k > 0$ ) does not approach classical Benfordness, even as a limit. On every bracket  $[10^{j-1}, 10^j - 1[$ , the leading digit is uniform. Therefore, taking  $k = 10^j - 1$  leads to a uniform (and not logarithmic) distribution for leading digits, whatever  $j$  might be.

The density  $g_k$  of  $\sqrt{X_k}$  is

$$g_k(x) = \frac{2x}{k}, \quad x \in ]0, \sqrt{k}]$$

and  $g_k(x) = 0$  otherwise. It is an increasing function on  $]-\infty, \sqrt{k}]$ , decreasing on  $[\sqrt{k}, +\infty[$  with maximum  $\frac{2}{\sqrt{k}} \rightarrow 0$  when  $k \rightarrow \infty$ . Theorem 2 applies, showing that  $X_k$  tends toward  $\sqrt{\cdot}$ -Benfordness in law. Now, theorem 3 below proves that  $X_k$  tends toward  $u$ -Benfordness, when  $u(x) = \pi x^2$ .

**Theorem 3.** *If  $X$  follows a uniform density on  $]0, k]$ ,  $\{\pi X^2\}$  converges in law toward uniformity on  $[0, 1[$  when  $k \rightarrow \infty$ .*

PROOF. Let  $X \sim U(]0, k])$ . The p.d.f.  $g$  of  $Y = \pi X^2$  is

$$g(x) = \frac{1}{2a\sqrt{x}}, \quad x \in ]0, a^2]$$

where  $a = k\sqrt{\pi}$ .

The c.d.f.  $G$  of  $Y$  is then

$$G(x) = \frac{\sqrt{x}}{a}, \quad x \in ]0, a^2].$$

Let now  $\delta \in ]0, 1[$ . Call  $P_\delta$  the probability that  $\{Y\} < \delta$ .

$$\sum_{j=0}^{\lfloor a^2 - \delta \rfloor} G(j + \delta) - G(j) \leq P_\delta \leq \sum_{j=0}^{\lfloor a^2 - \delta \rfloor + 1} G(j + \delta) - G(j)$$

$$\frac{1}{a} \sum_{j=0}^{\lfloor a^2 - \delta \rfloor} \sqrt{j + \delta} - \sqrt{j} \leq P_\delta \leq \frac{1}{a} \sum_{j=0}^{\lfloor a^2 - \delta \rfloor + 1} \sqrt{j + \delta} - \sqrt{j}$$

The square-root function being concave,

$$\sqrt{j + \delta} - \sqrt{j} \geq \frac{\delta}{2\sqrt{j + \delta}}$$

and, for any  $j > 0$ ,

$$\sqrt{j + \delta} - \sqrt{j} \leq \frac{\delta}{2\sqrt{j}}.$$

Hence,

$$\begin{aligned} \frac{\delta}{2a} \sum_{j=0}^{\lfloor a^2 - \delta \rfloor} \frac{1}{\sqrt{j + \delta}} &\leq P_\delta \leq \frac{1}{a} \left[ \sqrt{\delta} + \sum_1^{\lfloor a^2 - \delta \rfloor + 1} \frac{\delta}{2\sqrt{j}} \right] \\ \frac{\delta}{2a} \sum_{j=0}^{\lfloor a^2 - \delta \rfloor} \frac{1}{\sqrt{j + \delta}} &\leq P_\delta \leq \frac{\sqrt{\delta}}{a} + \frac{\delta}{2a} \sum_1^{\lfloor a^2 - \delta \rfloor + 1} \frac{1}{\sqrt{j}} \end{aligned}$$

$x \mapsto \frac{1}{\sqrt{x}}$  being decreasing,

$$\sum_{j=0}^{\lfloor a^2 - \delta \rfloor} \frac{1}{\sqrt{j + \delta}} \geq \int_{\delta}^{\lfloor a^2 - \delta \rfloor + 1 + \delta} \frac{1}{\sqrt{t}} dt \geq 2 \left[ \sqrt{\lfloor a^2 - \delta \rfloor + 1 + \delta} - \sqrt{\delta} \right]$$

and

$$\sum_1^{\lfloor a^2 - \delta \rfloor + 1} \frac{1}{\sqrt{j}} \leq \int_0^{\lfloor a^2 - \delta \rfloor + 1} \frac{1}{\sqrt{t}} dt \leq 2 \left[ \sqrt{\lfloor a^2 - \delta \rfloor + 1} \right].$$

So,

$$\frac{\delta}{a} \left[ \sqrt{\lfloor a^2 - \delta \rfloor + 1 + \delta} - \sqrt{\delta} \right] \leq P_\delta \leq \frac{\delta}{a} \left[ \sqrt{\lfloor a^2 - \delta \rfloor + 1} \right].$$

As a consequence, for any fixed  $\delta$ ,  $\lim_{a \rightarrow \infty} (P_\delta) = \delta$ , and  $\{\pi X^2\}$  converges in law to uniformity on  $[0, 1[$ .

#### 4.3.2. Exponential r.v.'s

Let  $X_\lambda$  be an exponential r.v. with p.d.f.  $f_\lambda(x) = \lambda \exp(-\lambda x)$  ( $x \geq 0, \lambda > 0$ ). Engel and Leuenberger [2003] demonstrated that  $X_\lambda$  tends toward the Benford's law when  $\lambda \rightarrow 0$ .

The p.d.f. of  $\sqrt{X_\lambda}$  is  $x \mapsto 2\lambda x \exp(-\lambda x^2)$ , which increases on  $]0, \frac{1}{2\lambda}]$  and then decreases. Its maximum is  $\exp(-\frac{1}{4\lambda})$ . Theorem 2 thus applies, showing that  $X_\lambda$  is  $\sqrt{\cdot}$ -Benford as a limit when  $\lambda \rightarrow 0$ .

Finally, theorem 4 below demonstrates that  $X_\lambda$  tends toward  $u$ -Benfordness for  $u(x) = \pi x^2$  as well.

**Theorem 4.** *If  $X \sim EXP(\lambda)$  (with p.d.f.  $f : x \mapsto \lambda \exp(-\lambda x)$ ), then  $Y = \pi X^2$  converges toward uniformity mod 1 when  $\lambda \rightarrow 0$ .*

PROOF. Let  $X$  be such a r.v.  $Y = \pi X^2$  has density  $g$  with

$$g(x) = \frac{\mu}{2\sqrt{x}} \exp(-\mu\sqrt{x}), \quad x \geq 0$$

where  $\mu = \frac{\lambda}{\sqrt{\pi}}$ . The  $Y$  c.d.f.  $G$  is thus, for all  $x \geq 0$

$$G(x) = 1 - e^{-\mu\sqrt{x}}.$$

Let  $P_\delta$  denote the probability that  $\{Y\} < \delta$ , for  $\delta \in ]0, 1[$ .

$$P_\delta = \sum_{j=0}^{\infty} \left[ e^{-\mu\sqrt{j}} - e^{-\mu\sqrt{j+\delta}} \right]$$

$x \mapsto \exp(-\mu\sqrt{x})$  being convex,

$$\delta \frac{\mu}{2\sqrt{j+\delta}} e^{-\mu\sqrt{j+\delta}} \leq e^{-\mu\sqrt{j}} - e^{-\mu\sqrt{j+\delta}}$$

for any  $j \geq 0$ , and

$$e^{-\mu\sqrt{j}} - e^{-\mu\sqrt{j+\delta}} \leq \delta \frac{\mu}{2\sqrt{j}} e^{-\mu\sqrt{j}}$$

for any  $j > 0$ . Thus

$$\delta \sum_{j=0}^{\infty} \frac{\mu}{2\sqrt{j+\delta}} e^{-\mu\sqrt{j+\delta}} \leq P_\delta \leq 1 - e^{-\mu\sqrt{\delta}} + \delta \sum_{j=1}^{\infty} \frac{\mu}{2\sqrt{j}} e^{-\mu\sqrt{j}}.$$

$x \longmapsto \frac{1}{\sqrt{x}} \exp(-\mu\sqrt{x})$  being decreasing,

$$\begin{aligned} \delta \sum_{j=0}^{\infty} \frac{\mu}{2\sqrt{j+\delta}} e^{-\mu\sqrt{j+\delta}} &\geq \delta \int_{\sqrt{\delta}}^{\infty} \frac{\mu}{2\sqrt{t}} e^{-\mu\sqrt{t}} dt \\ &\geq \delta \left[ -e^{-\mu\sqrt{t}} \right]_{\sqrt{\delta}}^{\infty} \\ &= \delta e^{-\mu\sqrt{\delta}}, \end{aligned}$$

and

$$\begin{aligned} 1 - e^{-\mu\sqrt{\delta}} + \delta \sum_{j=1}^{\infty} \frac{\mu}{2\sqrt{j}} e^{-\mu\sqrt{j}} &\leq 1 - e^{-\mu\sqrt{\delta}} + \delta \int_0^{\infty} \frac{\mu}{2\sqrt{t}} e^{-\mu\sqrt{t}} dt \\ &\leq 1 - e^{-\mu\sqrt{\delta}} + \delta \end{aligned}$$

The two expressions tend toward  $\delta$  when  $\mu \rightarrow 0$ , so that  $P_\delta \rightarrow \delta$ . The proof is complete.

#### 4.3.3. Absolute value of a normal distribution

To test the absolute value of a normal distribution  $X$  with mean 0 and variance  $10^8$ , we picked a sample of 2000 values and used the same procedure as for real data. It appears, as shown in Table 3, that  $X$  significantly departs from  $u$ -Benfordness with  $u = \log$  and  $u = \pi \cdot$ , but not with  $u = \sqrt{\cdot}$ .

	$\log(X)$	$\sqrt{X}$	$\pi X^2$
$U([0, k])$ $k \rightarrow \infty$	NO	YES	YES
$EXP(\lambda)$ $\lambda \rightarrow 0$	YES	YES	YES
$ \mathcal{N}(0, 10^8) $	14.49 (.000)	0.647 (.797)	28.726 (.000)

Table 3 — The table displays if uniform distributions, exponential distributions, and absolute value of a normal distribution, are  $u$ -Benford for different functions  $u$ , or not. The last line shows the results (and  $p$ -values) of the Kolmogorov-Smirnov tests applied to a 2000-sample. It could be read as "NO; YES; NO".

As we already noticed, the best shot when one is looking for Benford seems to be the square-root rather than  $\log$ .

## 5. Discussion

Random variables exactly conforming the Benford's classical law are rare, although many do roughly approach the law. Indeed, many explanations have been proposed for this approximate law to hold so often. These explanations involve complex characteristics, sometimes directly related to logarithms, sometimes through multiplicative properties.

Our idea — formalized in theorem 1 — is more simple and general. The fact that real data often are regular and scattered is intuitive. What we proved is an idea which has been recently expressed by Fewster [2009]: scatter and regularity are actually *sufficient* condition to Benfordness.

This fact thus provides a new explanation of Benford's law. Other explanations, of course, are acceptable as well. But it may be argued that some of the most popular explanations are in fact corollaries of our theorem. As we have seen when studying Pareto type II density, mixtures of distributions may lead to regular and scattered density, to which theorem 1 applies. Thus, we may argue that a mixture of densities is nearly Benford *because* it is necessarily scattered and regular. In the same fashion, multiplications of effects lead to Benford-like densities, but also (as the multiplicative central-limit theorem states) to regular and scattered densities.

Apart from the fact that our explanation is simpler and (arguably) more general, a good argument in its favor is that Benfordness may be generalized — unlike log-related explanations. Scale invariance or multiplicative properties are log-related. But as we have seen, Benfordness is not dependant on log, and can easily be generalized. Actually, it seems that square root is a better candidate than log. The historical importance of log-Benfordness is of course due to the implications in terms of leading digits which bears no equivalence with square-root.

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